*Journal of Statistical Physics, Vol. 54, No. 1/2, 1989*

# Local Structure Theory: Calculation on Hexagonal Arrays, and Interaction of Rule and Lattice

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*Received February 9, 1988; final August 9, 1988*

Local structure theory calculations[7] are applied to the study of cellular automata on the two-dimensional hexagonal lattice. A particular hexagonal lattice rule denoted (3422) is considered in detail. This rule has many features in common with Conway's *Life*. The local structure theory captures many of the statistical properties of this rule; this supports hypotheses raised by a study of *Life* itself.[6] As in *Life*, the state of a cell under (3422) depends only on the state of the cell itself and the sum of states in its neighborhood at the previous time step. This property implies that evolution rules which operate in the same way can be studied on different lattices. The differences between the behavior of these rules on different lattices are dramatic. The mean field theory cannot reflect these differences. However, a generalization of the mean field theory, the local structure theory, does account for the rule–lattice interaction.

**KEY WORDS:** Cellular automata; mean-field theory; statistical mechanics; chaos; dynamical systems; lattice gas.

## 1. INTRODUCTION

Cellular automata are discrete, deterministic dynamical systems with a variety of complex and chaotic behaviors.[14] A cellular automaton operates on a discrete, regular lattice of cells, each with a finite number of states. An interaction rule specifies the state of a cell at a given time in terms of the

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0022-4715/89/0100-0495$06.00/0 

states of nearby cells at a previous time. Our interest here is in a statistical description of cellular automaton behavior. That is, rather than thinking of a cellular automaton as acting on configurations of cell states on a lattice, we consider the action of an automaton on probability measures on the set of all possible configurations.

The local structure theory[7] is a model of the action of a cellular automaton on a probability measure. The idea is to replace a general measure by one whose structure is simple enough to allow analytical approximation of the cellular automaton's action on it. These simple measures are called finite block measures, since their properties are completely determined by the specification of the probabilities of contiguous sequences of cell states, called *blocks*, of a fixed finite size. The local structure operator, defined below, maps finite block measures to finite block measures in a way which approximates a cellular automaton map on general measures on the set of infinite lattice configurations.

There is a major difference between local structure theory in one dimension and in more than one dimension. In one dimension, there is a simple, explicit procedure for extending a given specification of probability of finite-sized blocks to a canonical shift-invariant measure on blocks of arbitrary size. In more than one dimension, it may be shown that there is no algorithm for deciding whether shift-invariant extension is possible at all. This is related to the undecidability of whether the plane can be tesselated with a given collection of polygons.[10,11]

Since cellular automata themselves are shift-invariant, we focus on their action on shift-invariant probability measures, In more than one dimension, it is not always possible to rigorously define shift-invariant measures in terms of finite-block probabilities. However, approximate extension formulas whose form is justified by heuristic arguments have been proposed.[6,12] In one case explored in detail,[6] these calculations provided an accurate estimate of the limit density of a prototypically complex rule on the two-dimensional square lattice, Conway's game of *Life*.[2]

Our aim here is to examine the application of the local structure theory to rules on the hexagonal lattice. There are several reasons for this: First, it will help establish whether success in predicting the limit density for *Life* was generic or fortuitous. Second, since the cells in the hexagonal lattice have fewer neighbors than those on the square lattice, exploration of higher order theory is possible. Third, these calculations may serve as a preamble to studies of hexagonal lattice gases.[4] Fourth, it will let us examine the ability of the local structure theory to account for effects of lattice connectivity on the nature of a cellular automaton rule.

A technique we will use to distinguish effects of lattice connectivity from those of cellular automaton rule structure is to define a sequence of

cellular automata on different lattices which have in some sense "the same" rule. The rules in the sequence are the same in that they have the same mean-field theory approximation. The differences between them are due to higher-order interactions due to lattice connectivity which can only be distinguished by higher-order local structure theory. We focus on a particular hexagonal rule which, though very similar to *Life* in rule structure, exhibits a very striking dependence on lattice connectivity which is *opposite*, in a sense, to that of *Life*.

## 2. EMPIRICAL RESULTS

### 1. Outer-Totalistic Rules

Rules may be said to be *outer-totalistic* if the state of a cell at the next time-step depends only on its own state, and the sum of the states of its neighbors.[9] We chose to study a class of outer-totalistic, nearest-neighbor rules on the hexagonal lattice in which each cell may have one of two states. Some of these results have the potential to support growth of configurations beyond their initial bounds. On the hexagonal lattice, such potential for growth implies that a "birth" (a cell which changes from 0 to 1) must be permitted if two or fewer neighbors are alive (in state 1). However, permitting a birth with only a single neighbor alive is likely to lead to uncontrollable growth. In the notation of Bays,[1] we thus limit ourselves to rules of the form $(xy22)$.[4] Typical limit configurations which arise from application of rules of the form $(xy22)$ to a random, 50 % occupied initial configuration are shown in Fig. 1. Three of the rules result in limit configurations of high density. Of these, two develop obvious correlation structure and one does not. The other three rules have limit configurations consisting of a sparse collection of stable structures. Rule (3422) has the greatest variety of stable structures, and was explored in more detail.

Rule (3422) was found to give rise to dozens of distinct periodic structures, with periodicities 2, 3, 4, 5, 10, 12, and 48. As in *Life* these structures were encountered in the evolution of simple starting configurations (such as triangles and hexagons). Random configurations evolved into sparse collections of these stable configurations. Some of these periodic structures transformed into rotated copies (rotation of $2\pi/3$) or mirror images at

[4] Bays' notation $(E_l E_u F_l F_u)$ denotes a rule in which the number of neighbors must be in the (inclusive) range $[E_l, E_u]$ for survivial, and in the range $[F_l, F_u]$ for birth. Conway's *Life* is denoted (2333).

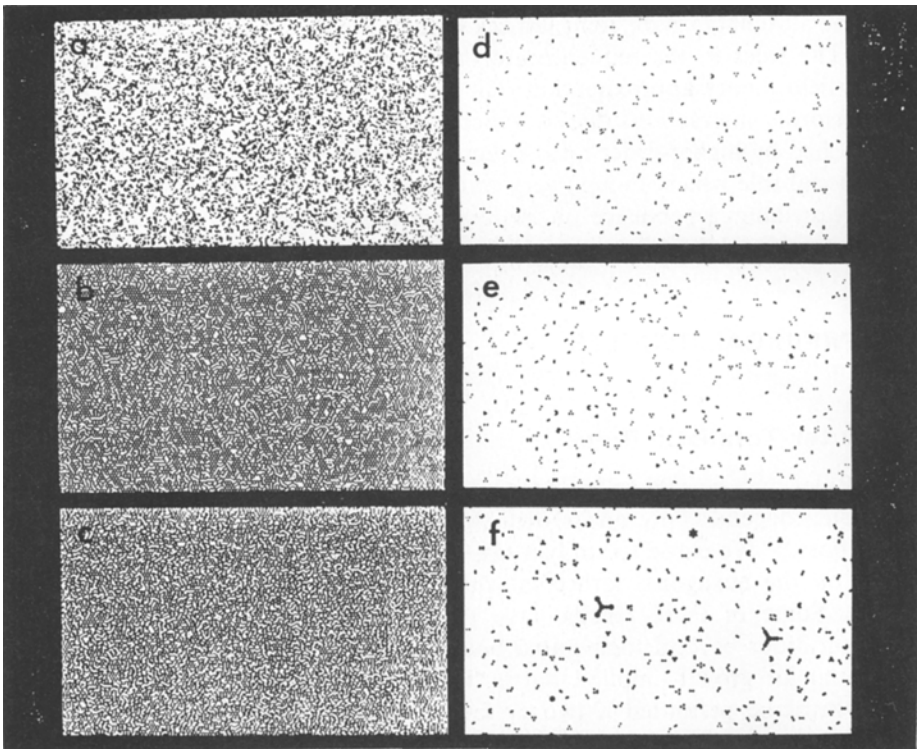


Fig. 1. Configurations at generation 300 under several hexagonal rules of the form (*xy*22). Random initial configurations. (a) (2222); (b) (2322); (c) (2422); (d) (3322); (e) (4422); (f) (3422).

appropriate submultiples of the period. Some periodic structures are shown in Fig. 2.

In addition to these spontaneously arising stable structures, rule (3422) also supports enclosures of arbitrary length and width (for example, Fig. 3). The enclosures are not destroyed by collision from the inside. Thus within such a barrier, (3422) may be considered to emulate a variety of one-dimensional rules. An example of such an emulation is shown in Fig. 4. One generation of the nearest-neighbor rule $90^{(14)}$ is emulated by two generations of rule (3422). A live cell in rule 90 is represented by a triplet of cells in (3422), and the ends of the enclosure are reflecting barriers.

Although the rich set of periodic structures suggest that (3422) is "complex," it is far from clear that it is computationally universal. To demonstrate that *Life* is computationally universal, Conway[2] made heavy use of gliders (structures that translate after some number of interactions of

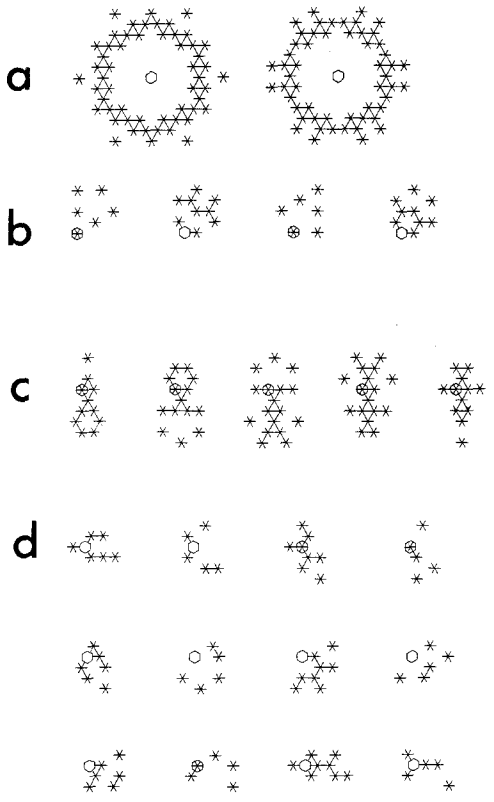


Fig. 2. Some stable structures of (3422) on the hexagonal lattice. (a) period 2; (b) period 4; (c) period 5; (d) period 12. A single hexagon is outlined for reference.

the rule). In rule (3422), gliders are either absent or extraorinarily rare: none were encountered in $10^6$ iterations of a Monte Carlo search algorithm which yielded gliders in *Life* at a rate of one per ten iterations.

## 2. Outer-Totalistic Rules in One-Dimension

Of the main features of *Life*, rich cycle structure and sparse limit configurations, but not the existence of gliders, are present in the rule (3422). To explore the extent to which these properties are due to the rule itself, or to lattice connectivity, we studied these same rules on a one-dimensional lattice (neighborhood radius 4 for *Life* and 3 for (3422)). On the one-dimensional lattice, random configurations under *Life* typically die, though they may evolve into simple stable configurations (prototypically, 01100110) (Fig. 5a). Rule (3422) in one dimension has more complex

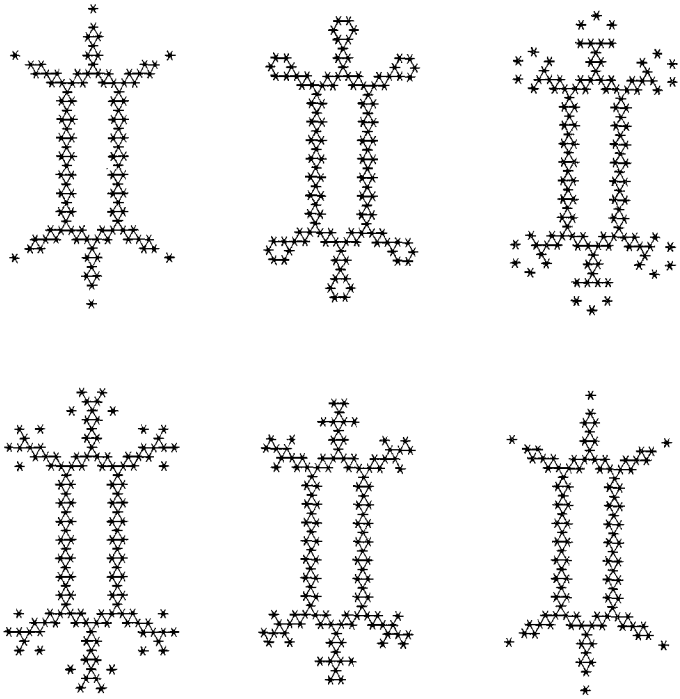

Fig. 3. An enclosure supported by rule (3422) (period 5).

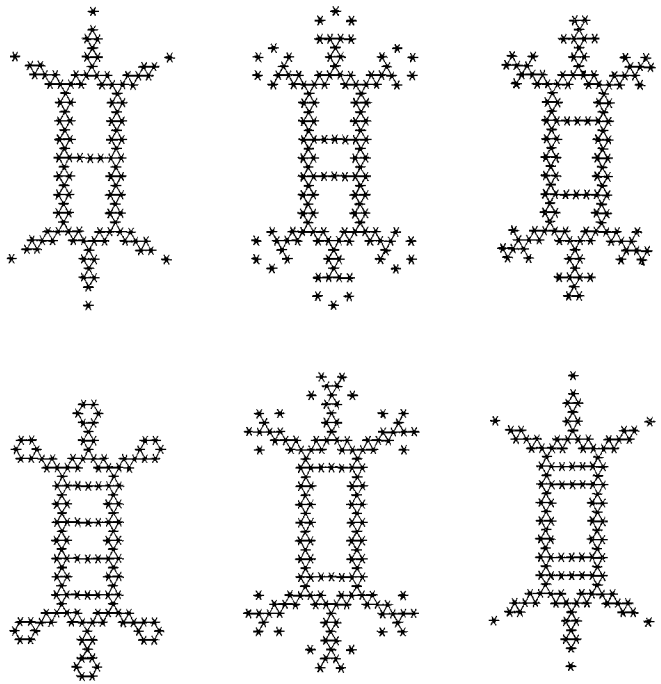

Fig. 4. An emulation within an enclosure by rule (3422) of the one-dimensional rule 90. Each successive picture represents two time-steps of rule (3422). This emulates one time-step of rule 90.

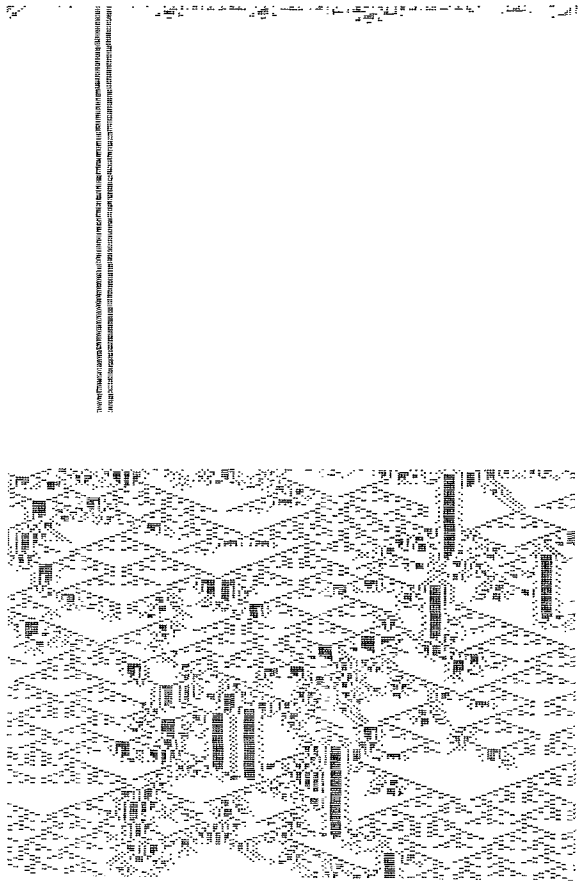

Fig. 5. Patterns generated by (a) *Life* and (b) (3422) on the one-dimensional lattice from a random initial start.

behavior (Fig. 5b). For example, gliders of period 1 (01010010) spontaneously form. They disintegrate when they collide with other structures. Interacting regions of the form ...00ii00jj00kk00... also develop. On these regions (3422) acts in a manner isomorphic to the linear, nearest-neighbor rule 90.

Thus *Life*, which in two dimensions supports a variety of complex structures, becomes rather simple in one dimension. Rule (3422) on the other hand, has complex behavior behavior in one dimension, including the generation of gliders, while in two dimensions it loses the ability to produce gliders.

The empirical limit densities for several outer-totalistic rules on the one-dimensional and two-dimensional hexagonal lattice are presented in

**Table I. Monte Carlo (MC) and Mean-Field (MF) Theory Limit Densities for Several Rules of the Form (*xy*22) on the One-Dimensional and Two-Dimensional Hexagonal Lattice. Initial Configurations Were Generated Pseudorandomly to Have Density 0.5**

| Rule | MF | MC-Hex | MC-1D |
|---|---|---|---|
| 2222 | 0.32 | 0.297 | 0.077 |
| 3322 | 0.27 | 0.017 | 0.282 |
| 2322 | 0.41 | 0.574 | 0.545 |
| 4422 | 0.19 | 0.023 | 0.283 |
| 2422 | 0.51 | 0.598 | 0.615 |
| 3422 | 0.31 | 0.021 | 0.274 |

Table I, along with mean-field calculations (see below). For these calculations, an initial density of 0.5 was used. Extensive empirical studies of rule (3422) indicate that its limit density does not depend appreciably on the initial density, provided that the initial density is in the range 0.2 to 0.8. For initial densities very close to 0. or to 1., the limit density is close to zero.

## 3. ANALYTIC APPROXIMATION OF STATISTICAL PROPERTIES

### 1. The Mean-Field Theory

The mean-field theory is a simple model of the statistical properties of a cellular automaton. The mean field theory is based on the assumption that the states of cells are uncorrelated with the states of other cells.[13] Under this assumption it is trivial to estimate the probability of any block of states in terms of the probability of a single state. The probability of a block is simply the product of the probabilities of the states of the cells in the block.

We assert that the probability that a cell has a particular state at a given time $t+1$ is the sum of the probabilities of the blocks that map to this state at time $t$. We need only consider blocks defined in the neighborhood of the cell of interest, because the cellular automaton rule is local. The neighborhood of a single cell under a nearest-neighbor celluar automaton on the hexagonal lattice, or a radius 3 rule on the one-dimensional lattice, is a collection of 7-cell blocks. Consider such an automaton, $\tau$, with two states per cell, labeled 0 and 1. If $B$ is a 7-block then $\tau(B)$ is

either a 0 or a 1. If $p_t$ is the probability of a 1 at time $t$ (the density at time $t$), then the mean-field theory gives the density at time $t+1$ as

$$p_{t+1} = \sum_B \tau(B)(p_t)^{\#1(B)}(1-p_t)^{\#0(B)} \tag{1}$$

where the sum is taken over all 7-blocks, and $\#0(B)$ and $\#1(B)$ are the number of 0's and 1's respectively in a block $B$. For a nearest neighbor cellular automaton on the square lattice, or a radius 4 rule on the one-dimensional lattice, the sum in Eq. 3 is taken over 9-blocks.

It is important to note that the mean-field theory does not distinguish between cellular automata which have the same evolution rule with the same number of neighbors, but are defined on different lattices. Thus, the fixed point of the mean-field theory equation for *Life*, 0.37, is an estimate for large-time density of both one- and two-dimensional versions of the rule. The mean-field fixed-point density is independent of initial density. Under one-dimensional *Life* the only cells in state 1 at large time are those which are part of a glider. Randomly chosen initial configurations rarely produce gliders, hence the large-time density for this rule is quite low. The large-time statistics depend on the initial statistics. If the initial density is 0.5, the large-time density is less than 0.05. For two-dimensional *Life* the large-time density is approximately 0.02, independent of initial density over a wide range. In the case of (3422), the mean-field fixed-point density is 0.30, while the empirical densities are 0.27 and 0.02 for the one- and two-dimensional versions, respectively. Thus, for these rules, the mean-field theory estimate of large-time density is reasonable only for (3422) on the one-dimensional lattice.

## 3.2. Local Structure Theory of One-Dimensional *Life* Analogs

In order to improve the statistical estimates provided by the mean-field theory of one dimensional cellular automata, Gutowitz *et al.*[7] introduced a generalization of the mean-field theory called the local structure theory. In the local structure theory, the probability of a block of states is estimated in terms of the probability of smaller blocks which it contains. An iteration process on block probabilities similar in form to Eq. 1 above is set up. The corresponding equations are now rational, rather than polynomial, in block probabilities. These equations will be discussed in more detail below. Here we present results for local structure theory applied to the one-dimensional analogs of *Life* and (3422).

We saw above that the mean-field theory yields an estimate of 0.37 for the large-time density of both one- and two-dimensional *Life*. Under the one-dimensional *Life* rule most initial configurations map to the 0 con-

figuration after a few time steps, so that the true large-time density for this rule is essentially 0. The fixed-point density for all orders of local structure theory 2–10 for one-dimensional *Life* is 0.

The results for one-dimensional (3422) are somewhat more complicated than the corresponding results for one-dimensional *Life*. In Fig. 6a, the fixed-point densities of each order of local structure theory 1 through 10 for (3422) are compared with the empirical large-time density. Intermediate orders of theory (orders 2–6) give less accurate estimates of the large-time density than the mean-field theory. However, still higher orders of theory provide increasingly accurate estimates.

Local structure theory of order greater than 1 supplies more information than just an estimate of the large-time density. Higher orders of

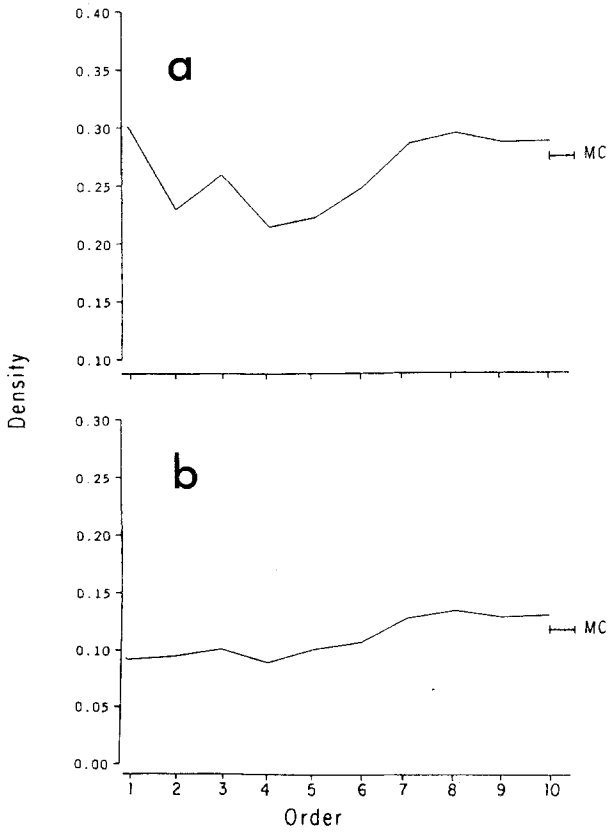


Fig. 6. (a) The fixed point density of each order of local structure theory 1–10 for (3422) on the one-dimensional lattice. MC indicates the empirical value of the large-time density for this rule. (b) As in (a), but for the block 11.

theory also supply estimates of the probability of blocks of larger size. When all of these estimates are taken into account, the accuracy of local structure theory estimates of block probabilities typically increases monotonically.[5] In Fig. 6b the large-time estimates of the probability of the 11 block for each order of theory (1–10) is compared to the empirical estimate of the large-time probability for this block. Here the theoretical estimates follow a more regular pattern than in the case of the density estimates.

Above we saw that the mean-field theory has two drawbacks. First, it does not provide good estimates of statistical properties of a cellular automaton. Second, it is not capable of reflecting the structure of the lattice on which the automaton operates. We have just seen that a one-dimensional local structure theory of sufficiently high order overcomes the first drawback of the mean-field theory. Below, we will develop local structure approximations for rules on the hexagonal lattice. We will see that these approximations overcome the second drawback of the mean-field theory as well.

## 3.3. Local Structure Theory on the Hexagonal Lattice

Local structure theory approximations for cellular automata on the square lattice have been studied previously.[6] It was found that a moderate order of theory provided good approximations to the statistical properties of *Life*. We will now review that theoretical development briefly, and then treat the case of the hexagonal lattice in more detail. Then we will apply the theory to rule (3422) on the hexagonal lattice.

The action of a cellular automaton, $\tau$, on a measure $\mu$ is defined by

$$\tau\mu(E) = \mu(\tau^{-1}(E)) \tag{2}$$

were $E$ is a $\mu$-measurable subset of configurations.[7] That is, the probability of $E$ under $\tau\mu$ is the sum of the probabilities under $\mu$ of the preimages of $E$ under $\tau$. In local structure theory, we represent $E$ by an interlocking cover of sets of a fixed size, and determine the preimages of the elements of this cover.

Explicitly, let a frame, $F$, be a finite subset of the lattice. A block, $B$, on $F$ is an assignment of states to some or all of the cells of $F$. Further, let $P_F$ be an assignment of probability to all blocks on $F$. We call $P_F$ a block probability function, and the probability assigned to a block $B$ will be denoted $P_F(B)$. A block probability function is required to satisfy the Kolmogorov consistency conditions.[3] That is, the probability assignment must be normalized, non-negative, and consistent: the sum of the probabilities of all blocks which contain another, smaller, block at a given

position is the same as the probability of the smaller block in that position. A block probability function is further required to be locally shift-invariant. This means that the probabilities assigned to all blocks on subframes within a given frame are the same, independent of *where* the subframe is placed within the given frame.

A finite block measure, $\mu_F$, is a measure which assigns probability to blocks on *all* frames in terms of the probabilities of blocks on a particular frame $F$. Probabilities of blocks on frames larger than $F$ must be determined by *extension* of a block probability function $P_F$. Extension on the hexagonal lattice is discussed below. Assume for the moment that extension of a block probability function is defined. Let $\sigma_F$ be a map from measures to measures such that $\sigma_F\mu$ is a measure which agrees with $\mu$ on blocks on $F$, and assigns probability to blocks on larger frames by extension. For each frame $F$, we define the local structure operator for a cellular automaton $\tau$ as

$$\Lambda_F(\tau)\,\mu(E) \equiv \sigma_F \tau \sigma_F \mu(E) \tag{3}$$

Intuitively, $\Lambda_F(\tau)$ is a map from measures to finite block measures which is an approximation to the map (2) associated with $\tau$. In the sense that $\sigma_F$ (which represents the extension of probabilities of blocks on $F$ to blocks on the entire lattice) becomes "close" to the identity as $F$ becomes large, $\Lambda_F(\tau)$ approaches $\tau$ as $F$ becomes large. Implementation of Eq. (3) hinges on the ability to effect the extension process $\sigma_F$. While one-dimensional extensions are exact and straight forward,[7] extensions on the hexagonal lattice are only approximate.

### 3.3.1. Extension of a block probability function on the hexagonal lattice

To implement $\Lambda_F(\tau)$ on the hexagonal lattice, it is necessary to estimate $P_{F'}$ given $P_F$, where $F'$ contains all cells which are neighbors of cell in $F$. An ideal estimate would: (1) satisfy the Kolmogorov consistency conditions; (2) be of maximum entropy among those $P_{F'}$ that are consistent with $P_F$; and (3) be shift-invariant. In one dimension, Bayesian extension[7] is guaranteed to satisfy these requirements. In more than one dimension, a shift-invariant extension may not *exist*, much less satisfy the Kolmogorov consistency conditions, or be of maximum entropy.[10,11]

We will obtain extensions by a simple, approximate method which is suggested by analogy to the one-dimension extension process. This extension method is based on a technique introduced in Ref. (12). In the extension process, large block probabilities are estimate by rational functions of the probabilities of small blocks which they contain. In accordance with the terminology introduced above, we will call this process Bayesian extension.

As pointed out by Schlijper ((Ref. 10) and personal communication), Bayesian extension suffers the drawbacks that the probability estimates obtained in this way need not: (1) be normalized; (2) preserve the symmetries of the lattice; (3) be locally shift-invariant; or (4) be of maximum entropy. Schlijper and Westerhof[12] compensated for potential asymmetries of the extension process by explicit symmetrization of the extended probabilities. Their method is presented below in the present context.

The goal is to define a block probability function on a frame $F'$ which is the preimage under the cellular automaton of a frame $F$. $F$ and $F'$ need not obey any lattice symmetry. Here the process is described in general. An explicit description of the actual extensions employed is given below. The strategy is as follows: We start with a block probability function $P_F$ on the frame $F$. For each block on $F'$, we begin matching blocks on $F$ one at a time with sub-blocks of the block on $F'$. A product of the probabilities of each new $F$-block is accumulated. Concurrently, a product of the probabilities of the intersection blocks is accumulated. Since we have assumed shift-invariance of the block probability function $P_F$, the intersection-block probabilities may be found by restriction of $P_F$ to the intersection frame using the Kolmogorov consistency conditions. Finally, the product of the $F$-block probabilities is divided by the product of the intersection-block probabilities. The intuitive reason for the division is that in the one-dimensional extension procedure it rigorously provides a maximum-entropy estimate.

In general, extension for $F$ to $F'$ may be done in stages: first from $F$ to $F_1$, then from $F_1$ to $F_2 \cdots$, and finally from $F_k$ to $F'$. In the extension processes described below, a new stage will always begin when block probabilities have been defined on a frame which respects some of the symmetries of the lattice. The reason is that asymmetries may have been introduced by the extension process to that point. This is compensated for by averaging the estimates over the appropriate lattice symmetries, and then normalizing. Extension continues with symmetrized block probabilities. New stages may also begin when probabilities have been defined on frames to be used as intersection frames in succeeding stages (see below).

Below, two special cases of this extension process are discussed in detail. Numerical studies based on these extensions will then be presented. We use the following notation for particular frames on the hexagonal lattice. Frames consisting of cells in a closest-packed arrangement are denoted with a single subscript. A single cell is denoted $F_1$. $F_2$ is a contiguous pair of cells, $F_3$ is a triangle of cells, and $F_7$ is a cell with its 6 nearest neighbors. Rectangular arrays of cells are denoted with two subscripts; e.g. $F_{(1\times 3)}$ is a single line three cells long. The basic frame of the mean field theory is $F_1$.

The two cases of local structure theory we will develop below are those in which the basic frame $F$ of $\Lambda_F(\tau)$ is $F_3$ and $F_7$. The entire extension process for these two cases is summarized in the notation described above in Tables II and III. In the following sections, the extensions will be described in words.

### 3.3.1.1. *The triangle approximation, $F_3$*

The collection of neighbors of cells in $F_3$ under a nearest neighbor cellular automaton will be denoted $F_{12}$. Hence, to implement $\Lambda_{F_3}(\tau)$, we must compute the probability of blocks on $F_{12}$ in terms of the probability of blocks on $F_3$. The extension from $F_3$ to $F_{12}$ will be presented in stages: $F_3$ to $F_5$, $F_5$ to $F_7$, and $F_7$ to $F_{12}$ (see Fig. 7). For a summary of the extension stages underlying the triangle approximation see Table II. Here the stages are described in detail.

(1) $F_3$ to $F_5$

The probability of a block on $F_5$ is estimated by the product of the probabilities of the thee triangles which fit into $F_5$, divided by the product

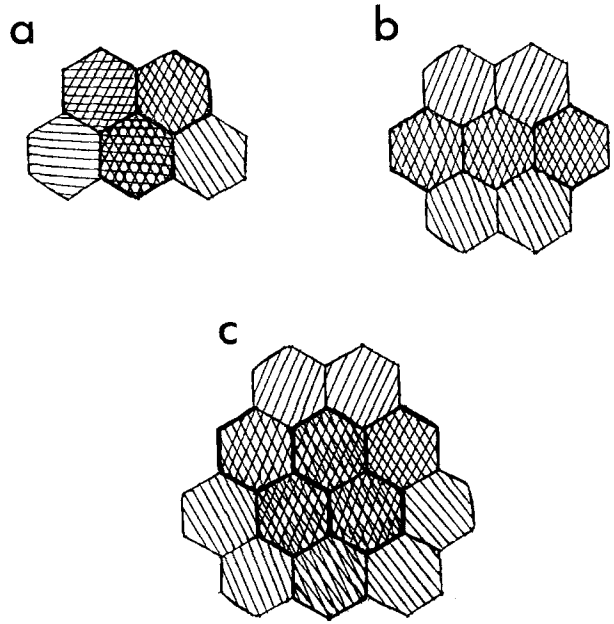


Fig. 7. (a) The extension of a block probability function on $F_3$ to a block probability function on $F_5$. (b) The extension of a block probability function on $F_5$ to a block probability function on $F_7$. (c) The extension of a block probability function on $F_7$ to a block probability function on $F_{12}$.

of the probabilities of the blocks on $F_2$ which occur as intersections of the triangles (Fig. 7a). The $F_2$ block probabilities are found by summation over the appropriate triangle probabilities.

(2) $F_5$ to $F_7$

We next estimate the probability of a block on $F_7$ as the product of blocks on the two $F_5$ frames which cover $F_7$, divided by the probability of their $(1 \times 3)$-block intersection (Fig. 7b). The probabilities of the $(1 \times 3)$-blocks can be obtained by partial summation of the $F_5$ probabilities just obtained.

Note that the extension shown in Fig. 7b has only two-fold symmetry, while $F_7$ has six-fold symmetry. Thus, a block probability measure of $F_3$ which respects the lattice may not result in symmetric assignment of probabilities to blocks on $F_7$. In order to recover a symmetry-preserving extension, the construction described above is performed in above in all possible orientations, and then the results are averaged.

(3) $F_7$ to $F_{12}$

Probabilities of blocks on $F_{12}$ are found by the overlap of three copies of $F_7$ as shown in Fig. 7c. The first two copies have an $F_{10}$ union and an $F_4$ intersection. The union of the remaining copy of $F_7$ with the $F_{10}$ frame just obtained is $F_{12}$, while the intersection of these two frames is $F_5$ of Fig. 7a. The $F_4$ and $F_5$ intersection block probabilities are found by restriction of the (symmetrized) $F_7$ probabilities from the previous stage of extension. This extension has no symmetry.

### 3.3.2. $\Lambda_{F_3}$

Let $b$ be a block on $F_3$, and $B$ a block on $F_{12}$. We estimate $P^{t+1}(b)$, the probability of $b$ at time $t+1$ in terms of $P^t(B)$, the probability of blocks $B$ at time $t$ by

$$P^{t+1}(b) = \sum_B \delta(b, \tau(B))\, P^t(B) \tag{4}$$

where the sum is taken over all blocks $B$ on $F_{12}$, and where $\delta$ is 1 if $\tau$ $(B)$ is identical to $b$, and 0 otherwise.

### 3.3.3. The hexagon approximation, $F_7$

The set of nearest neighbors of cells in $F_7$ is $F_{19}$. The process of obtaining estimates for probabilities of blocks on $F_{19}$ given probabilities of blocks on $F_7$ follows the pattern of extension described above for the triangle approximation. The first stage is the same as the third stage of the triangle approximation, in which block probability estimates on $F_{12}$ are

**Table II. Extension underlying the triangle approximation. Each stage contains a restriction and an extension. Extensions are indicated by an ↑ and restrictions by a ↓. Each stage uses the results of the previous stage. Some stages are terminated by a symmetrization step**

1. $F_3 \uparrow F_5$ $F_3 \downarrow F_2$
   covering frame: $F_3$; intersection frame: $F_2$
2. $F_5 \uparrow F_7$ $F_5 \downarrow F_{(1\times 3)}$
   covering frame: $F_5$; intersection frame: $F_{(1\times 3)}$
   Symmetrize $F_7$
3. $F_7 \uparrow F_{12}$ $F_7 \downarrow F_4$; $F_7 \downarrow F_5$
   covering frame: $F_7$; intersection frames: $F_4$, $F_5$
   Symmetrize $F_{12}$

found in terms of the probabilities of blocks on $F_7$. In the second stage, probabilities of blocks on $F_{19}$ are found by the overlap of three copies of $F_{12}$ (not shown). The first two copies have an $F_{16}$ union and an $F_8$ intersection. $F_8$ probabilities are found by restriction of $F_{12}$ probabilities. The union of the remaining copy of $F_{12}$ with the $F_{16}$ frame just obtained is $F_{19}$, while the intersection of these two frames is a particular $F_9$ frame. Block probabilities on the intersection frame are found by restriction of $F_{12}$ probabilities. This extension is again asymmetric, and must be symmetrized as above. The entire extension process from $F_7$ to $F_{19}$ is summarized in Table III.

The equations for the evolution of probabilities of $F_7$-blocks under a cellular automaton are the same in form as Eq. 4; the sum is taken over all blocks $B$ on $F_{19}$.

## 3.4. Numerical Studies of Rule (3422)

We studied the evolution of block probabilities under the cellular automaton (3422) with Monte Carlo techniques and local structure theory

**Table III. Extension Process Underlying the Hexagon Approximation. Notation as in Table 2**

1. $F_7 \uparrow F_{12}$ $F_7 \downarrow F_4$; $F_7 \downarrow F_5$
   covering frame: $F_7$; intersection frames: $F_4$, $F_5$
   Symmetrize $F_{12}$
2. $F_{12} \uparrow F_{19}$ $F_{12} \downarrow F_8$; $F_{12} \downarrow F_9$
   covering frame: $F_{12}$; intersection frames: $F_8$, $F_9$
   Symmetrize $F_{19}$

as described above. In Fig. 8, the evolution of the density under (3422) on the hexagonal lattice is shown. In this figure, Monte Carlo sampling is compared with local structure theory approximations based on $F_1$, $F_3$, and $F_7$. The initial state in all cases is unbiased and uncorrelated. For the first four generations, the various orders of theory cannot be distinguished. As correlations develop under continued applications of the rule, the mean field theory and the triangle approximation deviate from the empirical density. The hexagonal approximation continues to track the empirical density curve well, until roughly generation 16, at which point it too begins to deviate. Even at high-generation numbers, however, the hexagon approximation correctly captures the general trend of the Monte Carlo curve. While the empirical density is approximately 0.0218, the mean field and triangle approximations have limiting values of 0.31 and 0.29, respectively. The hexagon approximation has a limit density of 0, which, though incorrect, is a much better estimate of the true limit density than the lower order estimates.

The density is a crude statistic. To better understand the distinctions between the different orders of approximation, we also studied the evolution of several 7-cell hexagon probabilities. These probabilities follow a variety of time courses to equilibrium. Some have a local extremum at a low generation number, others are monotonic.

Three representative cases are shown in Fig. 9. Figure 9a depicts the evolution of the probability of the hexagon which has a solitary 1 on the periphery. The mean field and triangle approximation are only able to

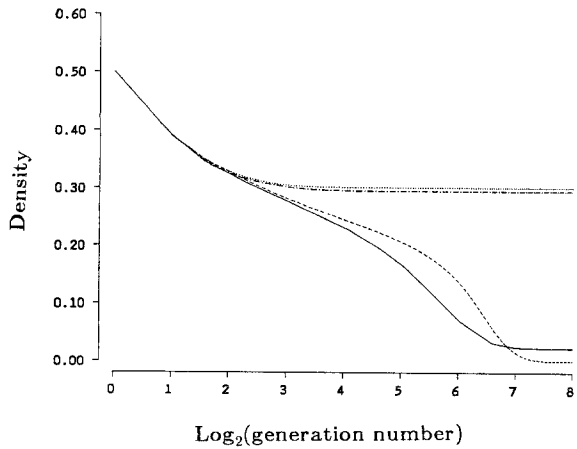


Fig. 8. Evolution of the density under (3422) on the hexagonal lattice. Monte Carlo results (solid line) are compared to estimates from $\Lambda_{F_1}$ (dotted line), $\Lambda_{F_3}$ (dot–dash), and $\Lambda_{F_7}$ (dashed line).

follow the rising phase of the empirical curve, while the hexagon approximation tracks the declining phase as well. This is a general observation: the larger the order of the theory, the longer it is able to follow the dynamics of a rule. In particular, the superiority of the triangle approximation over the mean-field theory is most evident at early times, as can be seen in Fig. 9b. The hexagon of all 0's (Fig. 9c) is the hexagon whose probability is least well-estimated by the low order theories. Schulman and Seiden[13] observed a similar phenomenon in their study of corrections to the mean field theory for *Life*. They found that if they

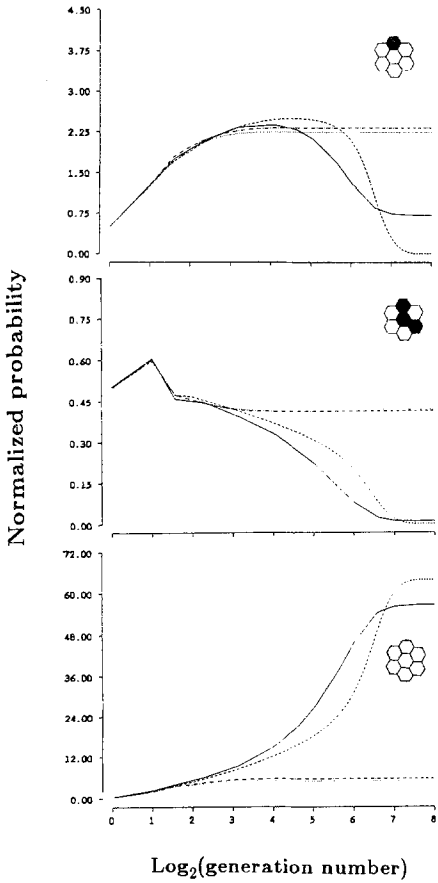


Fig. 9. Evolution of selected hexagon probabilities under (3422) on the hexagonal lattice. Monte-Carlo results (solid line) are compared to estimates from $\Lambda_{F_1}$ (dotted line), $\Lambda_{F_3}$ (dot–dash), and $\Lambda_{F_7}$ (dashed line). The normalized initial probability for each block is 0.5.

explicitly compensated for the "vacuum" their corrected mean field theory was in better accord with experiment. Here we found that the unaided hexagon approximation is sufficient to estimate the extent of the vacuum.

## 4. CONCLUSIONS

We developed an approximation scheme for cellular automaton on the two-dimensional hexagonal lattice. We saw moderate orders of approximation reasonably capture the statistical features of a particular complex automaton on that lattice.

This rule, (3422), was chosen for detailed study because it appears to share many of the features of *Life* on the two-dimensional square lattice. It too has a low limit density (about 0.02 in each case), and supports a wide variety of small, temporally periodic structures. The one crucial attribute of *Life* not possessed by hexagonal (3422) is the ability to produce gliders. We suggest that this distinction between *Life* and (3422) is due to an interaction between the nature of a cellular automaton rule and that of the lattice on which it operates.

To explore this suggestion, we studied (3422) on the line and the two-dimensional hexagonal lattice. We saw through empirical simulation that the same rule can exhibit very different behavior on lattices of different structure.

The mean-field theory fails to distinguish one lattice from another. Hence, the mean-field theory approximations of (3422) on the various lattices are identical. Higher order local structure theory *can* distinguish between lattices, and does so increasingly well as order increases. This is demonstrated by comparison of higher order local structure theory approximations of the one-dimensional and two-dimensional hexagonal lattice version of (3422).

We employed an extension of block probabilities on the hexagonal lattice based on a technique of Schlijper and Westerhof.[12] They considered the problem of estimating the critical temperature of the Ising model on the hexagonal lattice. They found that, using only extension of $P_{F_3}$ to $P_{F_7}$, they could estimate the critical temperature within 6 %, while the mean-field theory is only accurate to within 65 %. In the present study, we found a similar degree of improvement over mean-field theory in the estimation of statistical properties of cellular automata.

The results presented here have potential bearing on studies of lattice gas automata.[4] In principle, the method developed here may be applied directly to the lattice gas automaton on the hexagonal lattice. In practice, however, the large number of possible cell states in these rules prohibits direct application of the method. It may be possible to take advantage of

the conservation laws obeyed by latice gases to break the approximation into managable pieces.

It is clear why a lattice gas on the hexagonal lattice "works," while a lattice gas on the square lattice does not. The square lattice automaton conserves certain quantities not conserved by a physical gas. For more general automata, such as those studied here, it may not be possible to appeal to physical intuition to explain the differences in behavior of otherwise identical rules on different lattices. We hope that continued efforts along the lines pursued here will provide deeper intuitions.

## ACKNOWLEDGMENTS

This work was supported in part by an IBM postdoctoral fellowship, a Cornell Scholars award, NSF Grant DMS-8505442, and NIH grants EY 6871, EY1428 and NS877. We would like to thank M. Feigenbaum and B. Knight for discussions and support.

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